Trigonometry Revealed

remember the end of trig?

$$\sin 2t = 2 \sin t \cos t$$

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$\sin \frac{1}{2} t = \pm \sqrt{\frac{1}{2}(1 - \cos t)}$$

you had to take them
on faith. . .
No longer!

108

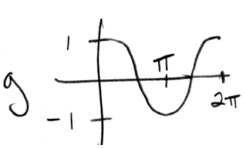
Now you can prove them

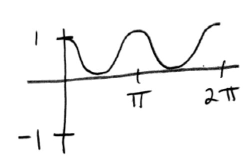
example

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$\left(\frac{e^{j\theta} + e^{-j\theta}}{2}\right)^2 = \frac{e^{j2\theta} + e^{-j2\theta} + e^0 + e^0}{4}$$

$$= \frac{\cos 2\theta}{2} + \frac{1}{2}$$

squaring 

yields 

109

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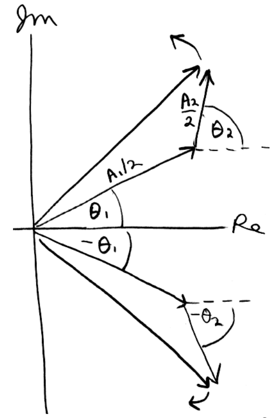
Superposition Revealed

The forward and backward spinning phasors within two sinusoids at the same frequency group independently.



Rotate as 2 rigid bodies

$$A_1\cos(\omega t+\theta_1) + A_2\cos(\omega t+\theta_2)$$

$\underline{\underline{is}}$ a sinusoid of frequency ω

$$\underbrace{\left[\frac{A_1}{2}e^{j\theta_1}+\frac{A_2}{2}e^{j\theta_2}\right]}e^{j\omega t} + \underbrace{\left[\frac{A_1}{2}e^{-j\theta_1}+\frac{A_2}{2}e^{-j\theta_2}\right]}e^{-j\omega t}$$

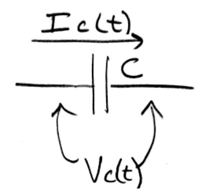
Single complex number and its complement completely describe its phase and amplitude.

110

Complex Impedance - Capacitor

Complex impedance Z replaces resistance R (which is real)

$$I_c(t) = C\frac{dV_c(t)}{dt}$$



if $V_c(t) = e^{j\omega t}$

then $I_c(t) = j\omega C e^{j\omega t}$

Complex Impedance using Ohms Law

$$Z_c = \frac{V_c(t)}{I_c(t)} = \frac{e^{j\omega t}}{j\omega C e^{j\omega t}} = \boxed{\frac{1}{j\omega C}} = -\frac{j}{\omega C}$$

111

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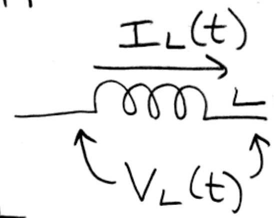
Complex Impedance - Inductor

Likewise for a coil

$$V_L(t) = L\,\frac{dI_L(t)}{dt}$$



if $I_L(t) = e^{j\omega t}$

then $V_L(t) = j\omega L\, e^{j\omega t}$

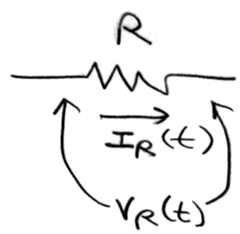
complex impedance

$$Z_L = \frac{V_L(t)}{I_L(t)} = \frac{j\omega L\, e^{j\omega t}}{e^{j\omega t}} = \boxed{j\omega L}$$

112

Complex Impedance - Resistor

what is complex impedance of resistor



$V_R(t) = e^{j\omega t}$

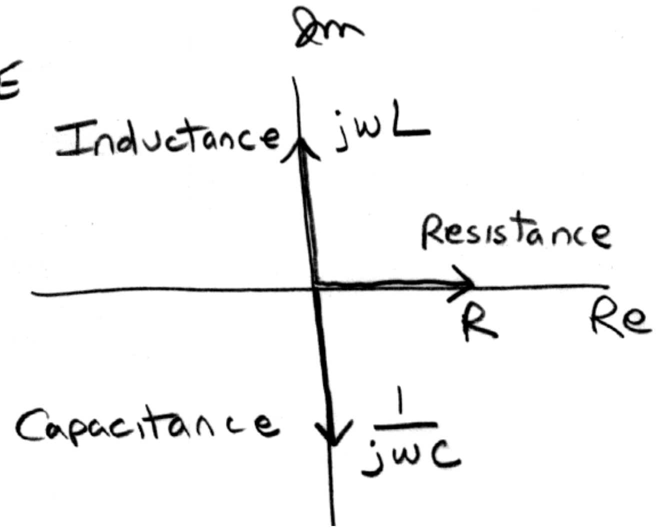
$I_R(t) = \dfrac{e^{j\omega t}}{R}$

$Z_R = \boxed{R}$ pure <u>real</u>

113

Impedance on the Complex Plane

COMPLEX IMPEDANCE Z



114

Taxonomy of Impedance

Complex Impedance

impedance Z $\xrightarrow{\text{real}}$ resistance $R = \frac{1}{\text{conductance } G}$

impedance Z $\xrightarrow{\text{imaginary}}$ reactance X

reactance X → capacitance $X_C = \frac{1}{j\omega C}$

reactance X → inductance $X_L = j\omega L$

$\frac{1}{Z}$ = "admittance"

"X" sometimes used if purely imaginary, or just "Z_C" and "Z_L"

115

Series Capacitors and Inductors

Series

$Z_1 + Z_2$

Two capacitors in series:

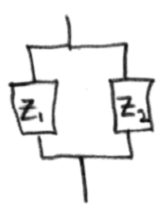
$$Z = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} = \frac{C_1 + C_2}{j\omega C_1 C_2} = \frac{1}{j\omega\left(\dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}\right)}$$

Two inductors in series:

$$Z = j\omega L_1 + j\omega L_2 = j\omega\left(L_1 + L_2\right)$$

Parallel Capacitors and Inductors

Parallel



$$\frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}}$$

Two capacitors in parallel:

$$Z = \frac{1}{j\omega C_1 + j\omega C_2} = \frac{1}{j\omega\left(C_1 + C_2\right)}$$

Two inductors in parallel:

$$Z = \frac{1}{\dfrac{1}{j\omega L_1} + \dfrac{1}{j\omega L_2}} = j\omega\left(\frac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2}}\right)$$

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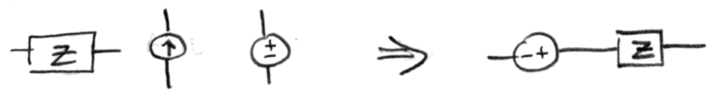
Same rules as DC circuits

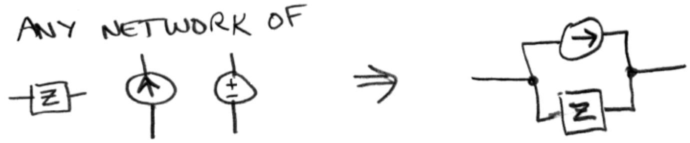
THEVENIN EQUIVALENT

ANY NETWORK OF

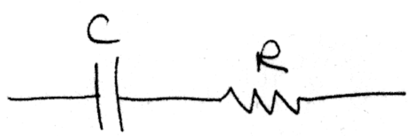


NORTON EQUIVALENT

ANY NETWORK OF



Impedance of a Passive Branch – RC circuit



$$Z = \frac{1}{j\omega C} + R = \frac{1 + j\omega \overbrace{RC}^{\text{time constant}}}{j\omega C}$$

$Z = R \Big|_{\omega \gg \frac{1}{RC}}$ Resistor dominates

$Z = \frac{1}{j\omega C} \Big|_{\omega \ll \frac{1}{RC}}$ Capacitor dominates

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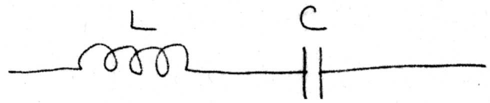
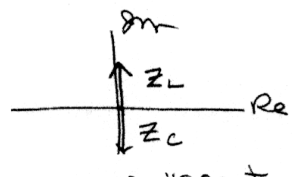
LC circuit - Resonance



$$Z = j\omega L + \frac{1}{j\omega C} = \frac{1-\omega^2 \overbrace{LC}^{\text{time constant}}}{j\omega C}$$

$$Z = 0 \;\Big|_{\omega = \frac{1}{\sqrt{LC}}} \quad \text{Resonance}$$

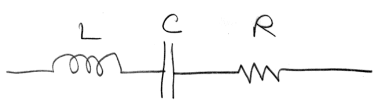
At resonance, impedances add to zero and cancel.



pure "reactance"

Analogous to spring and weight system – Energy in passed between magnetic and electric fields, as in electromagnetic wave.

Adding R to LC damps the ringing

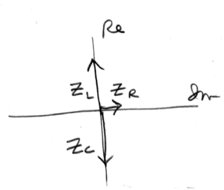


$$Z = j\omega L + \frac{1}{j\omega C} + R =$$

$$\frac{1-\omega^2 LC}{j\omega C} + R =$$

$$\underbrace{j\left(\frac{\omega^2 LC - 1}{\omega C}\right)}_{\text{pure imaginary}} + R$$

pure Real



can never equal zero

Like dragging your feet on the swing. Energy being passed from magnetic to electric field eventually dissipated by resistor as heat.

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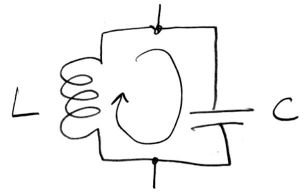
"Tank" Circuit

put coil and cap in loop



COOL FACT
speed of light

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

"inductance" and "capacitance" of free space

current sees no impedance at $\omega = \frac{1}{\sqrt{LC}}$

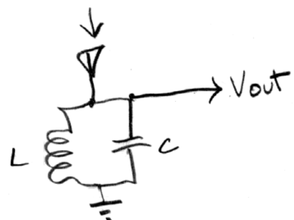
will resonate and absorb radio frequency (RF) energy

used in anti shoplifting tags

also pipe organs and microwave tubes.

122

"Tank" circuit first thing after antenna in radio receiver



$$Z = \frac{1}{\frac{1}{j\omega L} + \frac{1}{\frac{1}{j\omega C}}} = \frac{1}{\frac{1}{j\omega L} + j\omega C} =$$

$$\frac{j\omega L}{1 - \omega^2 LC} = \infty \Big|_{\omega = \frac{1}{\sqrt{LC}}}$$

- How can impedance be infinite through the parallel LC circuit when each of the components can pass current?
- At the resonant frequency the currents trying to pass from the antenna to ground are shifted 90° in opposite directions and thus are 180° out of phase and cancel.
- This "null point" is an example of how lenses work with light.

123

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